Power Series - Day 1

Worksheet 87

1) $\sum x^n$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x|$$

$|x| < 1$

$-1 < x < 1$

Test endpoints

$x = 1$: $\sum 1^n$ diverges by n^{th} term

$x = -1$: $\sum (-1)^n$ diverges by oscillation

$R = 1$

Interval: $-1 < x < 1$

2) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(4x+1)^{n+1}}{(-1)^n(4x+1)^n}\right| = |4x+1|$$

$|4x+1| < 1$

$-1 < 4x+1 < 1$

$-2 < 4x < 0$

$-\frac{1}{2} < x < 0$

Test endpoints

$x = -\frac{1}{2}$: $\sum_{n=0}^{\infty} (-1)^{2n}$ diverges by n^{th} term

$x = 0$: $\sum_{n=0}^{\infty} (-1)^n$ diverges by oscillation

$R = \frac{1}{4}$ Interval: $-\frac{1}{2} < x < 0$

3) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$ $c = \frac{2}{3}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{n+1}\cdot\frac{n}{(3x-2)^n}\right| = |3x-2|$$

$|3x-2| < 1$

$-1 < 3x-2 < 1$

$1 < 3x < 3$

$\frac{1}{3} < x < 1$

Test Endpoints

$x = \frac{1}{3}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges alternating harmonic

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent harmonic

$R = \frac{1}{3}$ Interval: $\frac{1}{3} \le x < 1$

4) $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$ $\quad c=-2$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+2)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n (x+2)^n} \right| = |x+2|$$

$$|x+2| < 1$$
$$-1 < x+2 < 1$$
$$-3 < x < -1$$

$R = 1$

Interval: $-3 < x \leq -1$

TEST ENDPOINTS

$x = -3$: $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$ — divergent harmonic

$x = -1$: $\sum \frac{(-1)^n}{n}$ — convergent alternating harmonic

5) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ $\quad c=1$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{(x-1)^n} \right| = |x-1|$$

$$|x-1| < 1$$
$$-1 < x-1 < 1$$
$$0 < x < 2$$

$R = 1$

Interval: $0 \leq x < 2$

TEST ENDPOINTS

$x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$

$\lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$

converges by AST

$x = 2$: $\sum \frac{1}{\sqrt{n}}$

diverges by p-series

6) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1$$

$R = \infty$

Interval: $(-\infty, \infty)$

7) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(n+1)!} \cdot \frac{n!}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n} \right| = 0 < 1$$

$R = \infty$

Interval: $(-\infty, \infty)$

8) $\sum_{n=0}^{\infty} \frac{n x^n}{4^n (n^2+1)}$ $c=0$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(n+1)\, x^{n+1}}{4^{n+1}((n+1)^2+1)} \cdot \frac{4^n (n^2+1)}{n\, x^n} \right| = \left| \frac{x}{4} \right|$$

$$\left| \frac{x}{4} \right| < 1$$

$$-1 < \frac{x}{4} < 1$$

$$-4 < x < 4$$

$R = 4$

Interval: $-4 \le x < 4$

TEST ENDPOINTS

$x = -4$

$\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

converges by AST

$x = 4$

$\sum_{n=0}^{\infty} \frac{n}{n^2+1}$

$\sum \frac{1}{n}$ diverges

$\lim_{n\to\infty} \left| \frac{n}{n^2+1} \cdot n \right| = 1$

diverges by LCT

9) $\sum_{n=0}^{\infty} n! (x-4)^n$ $c=4$

$\lim_{n\to\infty} \left(n! (x-4)^n \right) = \infty$

converges only when $x=4$

$R = 0$